International Journal of Theoretical Physics, Vol. 43, No. 1, January 2004 (© 2004)

Discrimination Among Quantum States

Masashi Ban[1]

Received September 1, 2003

It has recently been shown that quantum measurement is not always useful for discrimination among quantum states (K. Hunter (2003). *Physical Review A* **68**, 012306). This paper provides another proof of the necessary and sufficient condition for quantum measurement not to be useful and examines the condition by considering quantum measurement which discriminates between two spin-1/2 states in thermal equilibrium.

KEY WORDS: quantum measurement; state discrimination; optimization of detection probability.

1. INTRODUCTION

One of the most important problems in a quantum communication theory is to obtain an optimum quantum measurement that discriminates among n quantum states $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_n$ with minimum error probability (Helstrom, 1976; Nielsen and chuang, 2000). Such an optimum measurement is applied for a receiver of a quantum-state signal. Quantum measurement is mathematically described by positive operator-valued measure $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ (Helstrom, 1976; Holevo, 1982) which satisfy $\hat{X}_k \geq 0$ and $\Sigma_{k=1}^{n} \hat{X}_k = \hat{\text{I}}$, where $\hat{\text{I}}$ stands for an identity operator. The conditional probability $P(j|k)$ that the quantum state is inferred to be $\hat{\rho}_j$ if $\hat{\rho}_k$ is true is given by $P(j|k) = \text{Tr}[\hat{X}_j \hat{\rho}_k]$. When the prior probability of the quantum state $\hat{\rho}_k$ is π_k, where $\pi_k \geq 0$ and $\Sigma_{k=1}^{n} \pi_k = 1$, the average probability of correct detection is given by (Helstrom, 1976)

$$P_D = \sum_{k=1}^{n} P(k|k)\pi_k = \sum_{k=1}^{n} \pi_k \text{Tr}[\hat{X}_k \hat{\rho}_k] \tag{1}$$

and the average probability of error is $P_E = 1 - P_D$.

[1] Advanced Research Laboratory, Hitachi Ltd., Hatoyama, Saitama 350-0395, Japan; e-mail: m-ban@harl.hitachi.co.jp.

0020-7748/04/0100-0027/0 © 2004 Plenum Publishing Corporation

In the case of discrimination between two quantum states, the average probability of correct detection becomes

$$
\begin{aligned}
P_D &= \pi_1 + \mathrm{Tr}[(\pi_2\hat{\rho}_2 - \pi_1\hat{\rho}_1)\hat{X}_2] \\
&= \pi_2 + \mathrm{Tr}[(\pi_1\hat{\rho}_1 - \pi_2\hat{\rho}_2)\hat{X}_1].
\end{aligned} \tag{2}
$$

Here, let λ_j be the eigenvalue of the Hermitian operator $\pi_1\hat{\rho}_1 - \pi_2\hat{\rho}_2$ and let $|\lambda_j\rangle$ be the corresponding eigenstate. Then the maximum value of the average probability P_D of correct detection and the positive operator-valued measure, $\hat{X}_1$ and $\hat{X}_2$, describing the optimum quantum measurement are given by (Helstrom, 1976),

$$
P_D^{\max} = \pi_2 + \sum_{\lambda_j > 0} \lambda_j, \qquad \hat{X}_1 = \sum_{\lambda_j > 0} |\lambda_j\rangle\langle\lambda_j|, \qquad \hat{X}_2 = \hat{1} - \hat{X}_1. \tag{3}
$$

In general, optimum quantum measurement that discriminates among n quantum states $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_n$ with prior probabilities $\pi_1, \pi_2, \ldots, \pi_n$ is described by positive operator-valued measure $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ which satisfy the following conditions (Helstrom, 1976):

$$
\left(\sum_{k=1}^{n} \pi_k \hat{X}_k \hat{\rho} - \pi_j \hat{\rho}_j \right) \hat{X}_j = 0 \quad (j = 1, 2, \ldots, n), \tag{4}
$$

and

$$
\sum_{k=1}^{n} \pi_k \hat{X}_k \hat{\rho}_k - \pi_j \hat{\rho}_j \geq 0 \quad (j = 1, 2, \ldots, n). \tag{5}
$$

It is easy to check that the positive operator-valued measure $\hat{X}_1$ and $\hat{X}_2$ given by Eq. (3) satisfy these conditions. It is very difficult to obtain the positive operator-valued measure which satisfies Eqs. (4) and (5). The exact and analytic expression of the optimum positive operator-valued measure can be obtained only when quantum states have some symmetric properties (Ban *et al.*, 1997; Bernett, 2001; Eldar and Forney, 2001).

It has recently been shown that quantum measurement is not always useful for discrimination among quantum states (Hunter, 2003). In this paper, we give another proof of the necessary and sufficient condition for quantum measurement not to be useful and we investigate optimum quantum measurement that discriminates between two spin-1/2 states in thermal equilibrium. We find that if the strength of the thermal noise is greater than the value determined by the quantum states and prior probabilities, the quantum measurement is not useful and the guessing strategy becomes better. The meaning of the result is also considered.

2. CONDITION FOR QUANTUM MEASUREMENT NOT TO BE USEFUL

Recently it has been shown that quantum measurement is not always useful for discrimination among quantum states (Hunter, 2003). The necessary and sufficient condition under that quantum measurement becomes unuseful for discrimination among quantum states has been derived from Eqs. (4) and (5). The condition is given by

$$\pi_m \hat{\rho}_m - \pi_k \hat{\rho}_k \geq 0 \quad (k = 1, 2, \ldots, n) \tag{6}$$

for some value m. When this condition is satisfied, the positive operator-valued measurement that describes the optimum quantum measurement becomes

$$\hat{X}_m = \hat{1}, \qquad \hat{X}_k = 0 \qquad (k \neq m). \tag{7}$$

It is easy to see from Eq. (6) that the quantum state $\hat{\rho}_m$ is the most likely state, namely, $\pi_m \geq \pi_k$ (for all k). Hence if the inequality (6) is satisfied, the optimum strategy for the quantum state discrimination is to guess the most likely state.

The necessary and sufficient condition given by Eq. (6) can be derived directly from the average probability of correct detection P_D given by Eq. (1). Our task is to maximize the probability P_D by choosing $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ appropriately. First we assume that the condition given by Eq. (6) is satisfied. Using the relation $\sum_{k=1}^{n} \hat{X}_k = \hat{1}$, we can rewrite the detection probability P_D as

$$P_D = \pi_m + \sum_{\substack{k=1 \\ (k \neq m)}}^{n} \mathrm{Tr}[\hat{X}_k(\pi_k \hat{\rho}_k - \pi_m \hat{\rho}_m)]. \tag{8}$$

Since $\pi_m \hat{\rho}_m \geq \pi_k \hat{\rho}_k$ for all k, the every term in the summation on the right-hand side is negative or zero. Then to maximize the detection probability P_D, we should set the positive operator-valued measure as given in Eq. (7). In this case, the maximum value of the probability P_D is equal to the maximum prior probability π_m. Therefore the condition given by Eq. (6) is sufficient for the quantum measurement not to be useful for state discrimination.

Next we assume that the positive operator-valued measure given by Eq. (7) maximizes the average probability P_D of correct detection. Then the following inequality should hold for any other positive operator-valued measure $\hat{X}_1$, $\hat{X}_2, \ldots, \hat{X}_n$,

$$P_D^{\max} = \pi_m \geq \pi_m + \sum_{\substack{k=1 \\ (k \neq m)}}^{n} \mathrm{Tr}[\hat{X}_k(\pi_k \hat{\rho}_k - \pi_m \hat{\rho}_m)], \tag{9}$$

or equivalently

$$\sum_{\substack{k=1 \\ (k \neq m)}}^{n} \mathrm{Tr}[\hat{X}_k(\pi_k \hat{\rho}_k - \pi_m \hat{\rho}_m)] \leq 0. \tag{10}$$

Suppose that the inequality $\pi_k \hat{\rho}_k \leq \pi_m \hat{\rho}_m$ is violated for some value $k = m'$. In this case, when we set $\hat{X}_k = 0\,(k \neq m, m')$, the average probability of correct detection becomes

$$P_D = \pi_m + \mathrm{Tr}[(\pi_{m'} \hat{\rho}_{m'} - \pi_m \hat{\rho}_m)\hat{X}_{m'}], \tag{11}$$

with $\hat{X}_m + \hat{X}_{m'} = \hat{1}$. Then if $\hat{X}_{m'}$ is the projection operator onto the subspace corresponding to the positive eigenvalues of the operator $\pi_{m'} \hat{\rho}_{m'} - \pi_m \hat{\rho}_m$, the inequality $P_D > \pi_m$ holds, where the operator $\pi_{m'} \hat{\rho}_{m'} - \pi_m \hat{\rho}_m$ has positive eigenvalues by the assumption. This result is the contradiction to the fact that π_m is the maximum value of the average probability P_D of correct detection. Thus the condition given by Eq. (6) is necessary for the quantum measurement not to be useful for state discrimination.

In the case of the discrimination between two quantum states, the necessary and sufficient condition means

$$\pi_1 \hat{\rho}_1 - \pi_2 \hat{\rho}_2 \geq 0 \quad \text{or} \quad \pi_2 \hat{\rho}_2 - \pi_1 \hat{\rho}_1 \geq 0. \tag{12}$$

It is seen from Eq. (2) that the former yields $\hat{X}_2 = 0$ and the latter $\hat{X}_1 = 0$ to maximize the average probability of correct detection in the quantum state discrimination.

3. DISCRIMINATION BETWEEN THERMAL SPIN-1/2 STATES

We suppose that two quantum states to be discriminated by quantum measurement are spin-1/2 states in thermal equilibrium, which are given by density matrices,

$$\hat{\rho}_1 = \frac{1}{2}\hat{1} + \frac{1}{2}(1 - 2f)\hat{\sigma}_z, \tag{13}$$

$$\hat{\rho}_2 = \frac{1}{2}\hat{1} + \frac{1}{2}(1 - 2f)[\hat{\sigma}_z \cos\theta + \hat{\sigma}_x \sin\theta], \tag{14}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are the Pauli spin matrices and $\hat{1}$ is an identity matrix. In these equations, the parameter f represents the strength of the thermal noise, satisfying the inequality $0 \leq f \leq 1/2$. In the absence of the thermal noise, the density matrix $\hat{\rho}_1$ describes the eigenstate of $\hat{\sigma}_z$ with eigenvalue 1 and the density matrix $\hat{\rho}_2$ describes the spin state, the direction of which is inclined by angle θ from the spin state $\hat{\rho}_1$. The purity and the von Neumann entropy of the quantum state $\hat{\rho}_j\,(j = 1, 2)$ are given by $F = \mathrm{Tr}(\hat{\rho}_j)^2 = 1 - 2f(1 - f)$ and $S(\hat{\rho}_j) = -(1 - f)\log(1 - f) - f\log f$.

When the prior probability of the quantum state $\hat{\rho}_j$ is π_j, we need the eigenvalue and eigenstates of the Hermitian operator $\pi_2 \hat{\rho}_2 - \pi_1 \hat{\rho}_1$ to obtain the optimum quantum measurement which attains the maximum value of the average probability of correct detection. When the thermal noise f and the prior probability

$\pi_k (k = 0, 1)$ satisfy the inequality

$$f < \frac{1}{2}\left[1 - \frac{|\pi_1 - \pi_2|}{\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}\right], \tag{15}$$

the maximum value of the average probability of correct detection becomes

$$P_D^{\max} = \frac{1}{2}[1 + (1 - 2f)\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}], \tag{16}$$

and the positive operator-valued measure which describes the optimum measurement is given by

$$\hat{X}_1 = \frac{1}{2}\hat{1} + \frac{(\pi_1 - \pi_2 \cos\theta)\hat{\sigma}_z - (\pi_2 \sin\theta)\hat{\sigma}_x}{2\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}, \tag{17}$$

$$\hat{X}_2 = \frac{1}{2}\hat{1} - \frac{(\pi_1 - \pi_2 \cos\theta)\hat{\sigma}_z - (\pi_2 \sin\theta)\hat{\sigma}_x}{2\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}. \tag{18}$$

On the other hand, when the following inequality holds,

$$f \geq \frac{1}{2}\left[1 - \frac{|\pi_1 - \pi_2|}{\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}\right], \tag{19}$$

the maximum value of the average probability of correct detection and the optimum positive operator-valued measure become

$$P_D^{\max} = \max[\pi_1, \pi_2], \tag{20}$$

and

$$\hat{X}_1 = \theta(\pi_1 - \pi_2)\hat{1}, \qquad \hat{X}_2 = \theta(\pi_2 - \pi_1)\hat{1}, \tag{21}$$

where $\theta(x)$ is a step function defined as $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. This result means that if the thermal noise is large, we should always infer the quantum state which has the larger prior probability, regardless of the measurement outcome. In this case, the quantum measurement is not useful for the discrimination between the two quantum states.

We now investigate the meaning of the condition given by Eq. (15) or (19). We denote as ΔP the difference between the correct detection probability and the error probability in the quantum measurement described by Eqs. (17) and (18). Then ΔP is given by

$$\Delta P = (1 - 2f)\sqrt{1 - 4\pi_1\pi_1 \cos^2(\theta/2)}. \tag{22}$$

Furthermore, the difference ΔQ between the correct detection probability and the error probability in the quantum measurement described by Eq. (21) becomes

$$\Delta Q = |\pi_1 - \pi_2|. \tag{23}$$

Thus the condition given by Eq. (15) is equivalent to the inequality,

$$\Delta P > \Delta Q. \tag{24}$$

This result means the trivial fact that we should perform the quantum measurement which yields the smaller error probability.

When we perform the quantum measurement described by Eqs. (17) and (18), the posterior probability $P(j|k)$ that the quantum state was $\hat{\rho}_j$ if the quantum measurement yields the outcome k is calculated by the Bayes law, which is given by

$$P(1|1) = \pi_1 \frac{1 + A_1}{1 + B}, \qquad P(1|2) = \pi_1 \frac{1 - A_1}{1 + B}, \tag{25}$$

$$P(2|1) = \pi_2 \frac{1 + A_2}{1 + B}, \qquad P(2|2) = \pi_2 \frac{1 + A_2}{1 - B}, \tag{26}$$

where the parameters A_1, A_2 and B are given by

$$A_1 = (1 - 2f) \frac{\pi_1 - \pi_2 \cos\theta}{\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}, \tag{27}$$

$$A_2 = (1 - 2f) \frac{\pi_2 - \pi_1 \cos\theta}{\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}, \tag{28}$$

$$B = (1 - 2f) \frac{\pi_1 - \pi_2}{\sqrt{1 - 4\pi_1\pi_2 \cos^2(\theta/2)}}, \tag{29}$$

which satisfy the relation $\pi_1 A_1 - \pi_2 A_2 = B$. The condition that the most likely state before the quantum measurement is also the most likely state after the quantum measurement is represented by

$$\pi_j \geq \pi_k \to P(j|l) \geq P(k|l) \quad (l = 1, 2). \tag{30}$$

It is found that this condition is equivalent to Eq. (19). Thus, the condition that quantum measurement is not useful for the discrimination between the quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ is equivalent to the condition that the most likely state remains most likely after the quantum measurement.

The quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ considered in this section also appears in the output port of the quantum depolarizing channel $\hat{\mathscr{L}}$ when the input states are $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = |\theta\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$, where the quantum depolarizing channel $\hat{\mathscr{L}}$ is given by

$$\hat{\mathscr{L}}\hat{\rho} = (1 - p)\hat{\rho} + \frac{1}{2} p\hat{1} \quad (0 \leq p \leq 1). \tag{31}$$

The parameter p is related to f in Eqs. (13) and (14) by $p = 2f$.

4. CONCLUDING REMARKS

In this note, we have considered the quantum measurement which discriminates among quantum states. We have provided another proof of the necessary and sufficient condition under that quantum measurement is not useful for quantum-state discrimination, which was initiated by Hunter (Hunter, 2003). Using the two spin-$1/2$ states in thermal equilibrium, we have examined the condition.

REFERENCES

Ban, M., Kurokawa, K., Momose, R., and Hirota, O. (1997). Optimum measurements for dicrimination among symmetric quantum states and parameter estimation. *International Journal of Theoritical Physics* **36**, 1269–1288.

Bernett, S. (2001). Minimum-error discrimination between multiply symmetric states. *Physical Review A* **64**, 030303(R).

Eldar, Y. and Forney, G., Jr. (2001). On quantum detection and the square-root measurement. *IEEE Transactions on Information Theory* **IT-47**, 858–872.

Helstrom, C. (1976). Quantum Detection and Estimation Theory, Academic Press, New York.

Holevo, A. (1982). Probabilistic and Statistical Aspects of Quantum Theory, North-Holland, Amsterdam.

Hunter, K. (2003). Measurement does not always aid state discrimination. *Physical Review A* **68**, 012306.

Nielsen, M. and Chuang, I. (2000). Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, UK.